ON THE COEFFICIENTS IN CERTAIN ASYMPTOTIC FACTORIAL EXPANSIONS. I

BY

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Section 1: *Introduction.*

If s and r denote integers $\geqq 0$ with $s \leqq r$, then it is easy to show by means of Stirling's formula that the function

$$g(z)=\frac{\Gamma(z-\sigma_1)\,\Gamma(z-\sigma_2)\ldots\Gamma(z-\sigma_s)}{\Gamma(z-\varrho_0)\,\Gamma(z-\varrho_1)\ldots\Gamma(z-\varrho_r)}, \tag{1}$$

where $\sigma_1, \ldots, \sigma_s, \varrho_0, \ldots, \varrho_r$ are independent of z, possesses for large $|z|$ in the sector

$$-\pi+\epsilon< \arg z<\pi-\epsilon, \text{ when } \epsilon>0,$$

an asymptotic expansion of the form

$$g(z)=\alpha^{\alpha z}\left\{\sum_{m=0}^{M-1}\frac{c_m}{\Gamma(\alpha z+\beta+m)}+O\left(\frac{1}{\Gamma(\alpha z+\beta+M)}\right)\right\}. \tag{2}$$

Here M denotes an arbritrary integer $\geqq 0$ independent of z and

$$\alpha=r+1-s>0;\ \beta=\sigma_1+\ldots+\sigma_s-\varrho_0-\ldots-\varrho_r+\tfrac{1}{2}s-\tfrac{1}{2}r.$$

Mr. T. D. Riney [1]) has found a recurrence relation between the coefficients c_m in the case that the numbers $\varrho_0, \varrho_1, \ldots, \varrho_r$ are distinct. In that case the rational function

$$\psi(z)=\frac{(z-\sigma_1)(z-\sigma_2)\ldots(z-\sigma_s)}{(z-\varrho_0)(z-\varrho_1)\ldots(z-\varrho_r)} \tag{3}$$

possesses a partial fraction expansion of the form

$$\psi(z)=\sum_{\varrho}\frac{D(\varrho)}{z-\varrho} \tag{4}$$

extended over the poles ϱ of $\psi(z)$; the numerators $D(\varrho)$ are constant. Put for each positive integer h, for $a>0$ and for each b

$$p(h, a, b, \psi)=a\sum_{\varrho}D(\varrho)\frac{\Gamma(a\varrho+b+h-1)}{\Gamma(a\varrho+b)}; \tag{5}$$

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[1]) T. D. Riney, On the coefficients in asymptotic factorial expansions, Proceedings of the American Mathematical Society, 1956.

this notation indicates that the number p depends on h, a, b and upon the choice of the function $\psi(z)$ which has the partial fraction expansion (4). The recurrence relation between the coefficients c_m obtained by Mr. T. D. Riney can be written in the form

$$(6) \qquad c_m = -\frac{1}{m\,\alpha^{\alpha+1}} \sum_{n=0}^{m-1} c_n\, p(m+1+\alpha-n,\, \alpha,\, \beta+n,\, \psi).$$

In a similar way we shall find, if $s \leqq r$ and if the numbers $\varrho_0, \ldots, \varrho_r$ are distinct,

$$(7) \qquad (g(z))^{-1} = \alpha^{-\alpha z}\Big\{\sum_{m=0}^{M-1} \gamma_m\, \Gamma(\alpha z+\beta-m) + O\,(\Gamma(\alpha z+\beta-M))\Big\}.$$

The coefficients γ_m satisfy the recurrence relation

$$(8) \qquad \gamma_m = \frac{1}{m\,\alpha^{\alpha+1}} \sum_{n=0}^{m-1} \gamma_n\, q(m+1+\alpha-n,\, \alpha,\, \beta+n,\, \psi),$$

where

$$(9) \qquad q\,(h,\, a,\, b,\, \psi) = a \sum_{\varrho} D(\varrho)\, \frac{\Gamma(a\varrho+b)}{\Gamma(1+a\varrho+b-h)}.$$

As I have already remarked, Mr. T. D. Riney has deduced his recurrence relation (6) under the assumption that the numbers $\varrho_0, \varrho_1, \ldots, \varrho_r$ are distinct. He notices however that this condition involves no real loss of generality, since the result, obtained in the case that $\varrho_0, \varrho_1, \ldots, \varrho_r$ are distinct, yields, by passage to the limit as two or more ϱ values tend to equality, the corresponding result for the case that the numbers $\varrho_0, \varrho_1, \ldots, \varrho_r$ are not all distinct. In other words: if the numbers $\varrho_0, \varrho_1, \ldots, \varrho_r$ are not all distinct, then the recurrence relation (6) remains true, if we replace each coefficient by the corresponding limit. That is true, but the calculation of this limit is so difficult, that I prefer to treat this case in a direct way. To that end I put

$$\phi_1(v,\, h) = 1 \quad \text{and} \quad \phi_k(v,\, h) = \sum_{0<n_1<n_2<\ldots<n_{k-1}<h} \{(v+n_1-1)\,(v+n_2-1)\ldots(v+n_{k-1}-1)\}^{-1}$$

for each integer $k \geqq 2$. If we allow the case that two or more numbers ϱ are equal, we find for the function $\psi(z)$ defined in (3) a partial fraction expansion of the form

$$\psi\,(z) = \sum_{\varrho,k} \frac{D(\varrho, k)}{(z-\varrho)^k},$$

extended over the poles of $\psi(z)$, and over positive integers k which are at most equal to the multiplicity of the pole ϱ. Now I put for each positive integer h, for $a > 0$ and for each b

$$(10) \qquad p\,(h,\, a,\, b,\, \psi) = \sum_{\varrho,k} a^k\, D\,(\varrho,\, k)\, \phi_k\,(a\varrho+b,\, h)\, \frac{\Gamma(a\varrho+b+h-1)}{\Gamma(a\varrho+b)}$$

and

$$(11)\qquad q(h, a, b, \psi) = \sum_{\varrho, k} (-)^{k-1} a^k D(\varrho, k)\, \phi_k(1-a\varrho-b, h)\, \frac{\Gamma(a\varrho+b)}{\Gamma(1+a\varrho+b-h)}.$$

In the special case that the numbers $\varrho_0, \varrho_1, \ldots, \varrho_r$ are distinct, these formulas assume the form indicated in (5) and (9).

Not only in the case that the numbers $\varrho_0, \varrho_1, \ldots, \varrho_r$ are distinct, but also in the case that two or more ϱ values are equal, the functions $g(z)$ and $(g(z))^{-1}$ possess the asymptotic expansions (2) and (7), where the coefficients c and γ_m satisfy for $m \geqq 1$ the recurrence relations (6) and (8).

As an example I consider the function $g^2(z)$, where $g(z)$ is defined by (1), where $s \leqq r$ and where the numbers $\varrho_0, \ldots, \varrho_r$ are supposed to be distinct. Then $\psi^2(z)$,where $\psi(z)$ is defined by (3) has the partial fraction expansion

$$\psi^2(z) = \sum_{\varrho} \frac{D^2(\varrho)}{(z-\varrho)^2} + \sum_{\varrho} \frac{E(\varrho)}{z-\varrho},$$

where the sums are extended over the numbers $\varrho_0, \varrho_1, \ldots, \varrho_r$ and where $D(\varrho)$ has the same value as in (4), namely

$$D(\varrho) = \frac{(\varrho-\sigma_1)(\varrho-\sigma_2)\ldots(\varrho-\sigma_s)}{\prod\limits_{\substack{h=0\\ \varrho_h \neq \varrho}}^{r} (\varrho-\varrho_h)},$$

whereas

$$E(\varrho) = 2D^2(\varrho) \left\{ \frac{1}{\varrho-\sigma_1} + \ldots + \frac{1}{\varrho-\sigma_s} - \sum_{\substack{h=0\\ \varrho_h \neq \varrho}}^{r} \frac{1}{\varrho-\varrho_h} \right\};$$

if ϱ is equal to at least one of the numbers $\sigma_1, \sigma_2, \ldots, \sigma_s$, then the right hand side denotes zero. The formulas (10) and (11) yield

$$p(h, a, b, \psi^2) = a^2 \sum_{\varrho} D^2(\varrho) \frac{\Gamma(a\varrho+b+h-1)}{\Gamma(a\varrho+b)} \sum_{n=1}^{h-1} \frac{1}{a\varrho+b+n-1} + \\ + a \sum_{\varrho} E(\varrho) \frac{\Gamma(a\varrho+b+h-1)}{\Gamma(a\varrho+b)}$$

and

$$q(h, a, b, \psi^2) = -a^2 \sum_{\varrho} D^2(\varrho) \frac{\Gamma(a\varrho+b)}{\Gamma(1+a\varrho+b-h)} \sum_{n=1}^{h-1} \frac{1}{-a\varrho-b+n} + \\ + a \sum_{\varrho} E(\varrho) \frac{\Gamma(a\varrho+b)}{\Gamma(1+a\varrho+b-h)}.$$

Application of the above results obtained in (2) and (7) gives

$$g^2(z) = \alpha^{\alpha z} \left\{ \sum_{m=0}^{M-1} \frac{c_m}{\Gamma(\alpha z+\beta+m)} + O\left(\frac{1}{\Gamma(\alpha z+\beta+M)}\right) \right\}$$

and

$$g^{-2}(z) = \alpha^{-\alpha z} \left\{ \sum_{m=0}^{M-1} \gamma_m \Gamma(\alpha z+\beta-m) + O\left(\Gamma(\alpha z+\beta-M)\right) \right\},$$

where

$$\alpha = 2r + 2 - 2s; \qquad \beta = 2(\sigma_1 + \ldots + \sigma_s) - 2(\varrho_0 + \ldots + \varrho_r) + s - r - \tfrac{1}{2},$$

where the coefficients c_m and γ_m satisfy for $m \geqq 1$ the recurrence relations

$$c_m = -\frac{1}{m\alpha^{\alpha+1}} \sum_{n=0}^{m-1} c_n\, p(m+1+\alpha-n, \alpha, \beta+n, \psi^2)$$

and

$$\gamma_m = \frac{1}{m\alpha^{\alpha+1}} \sum_{n=0}^{m-1} \gamma_n\, q(m+1+\alpha-n, \alpha, \beta+n, \psi^2).$$

The proof given by Mr. T. D. Riney for the recurrence relation between the coefficients c_m is based on the fact that the function $g(z)$ satisfies the functional equation

$$\frac{g(z+1)}{g(z)} = \frac{(z-\sigma_1)\ldots(z-\sigma_s)}{(z-\varrho_0)\ldots(z-\varrho_r)} = \sum_{\varrho} \frac{D(\varrho)}{z-\varrho}.$$

This functional equation does not imply that $g(z)$ possesses an asymptotic expansion of the form (2), since the functional equation remains true if $g(z)$ is multiplied by a periodic function with period 1. In order to define by means of the functional equation the function $g(z)$, apart from a constant factor, we must add a certain condition of smoothness at infinity. In this paper I choose the condition that there exist three numbers $\varkappa$, μ and ν with the following property: if z is variable in such a way that $\operatorname{Im} z$ remains constant and that $\operatorname{Re} z$ approaches infinity, then

$$g(z)\, e^{-\varkappa z}\, z^{-\mu z-\nu}$$

ends to a finite limit $L \neq 0$ which is independent of the constant number $\operatorname{Im} z$. Then the function $g(z)$ is defined, apart from a constant factor, by this condition of smoothness and by the given functional equation.

In this paper I treat the following general problem. Suppose that a function $f(z)$, defined in a sector

$$-\pi + \epsilon < \arg z < \pi - \epsilon,$$

where $\epsilon > 0$, satisfies the condition of smoothness formulated above and assume that $\frac{f(z+1)}{f(z)}$ possesses in that sector for large $|z|$ an asymptotic expansion of the form

$$\frac{f(z+1)}{f(z)} \backsim \psi_0(z) + \psi_1(z) + \ldots, \tag{12}$$

where $\psi_0(z)$, $\psi_1(z)$, ... denote rational functions of z which tend to zero as $|z|$ approaches infinity. We shall see that the functions $f(z)$ and $(f(z))^{-1}$ possess asymptotic expansions similar to those obtained in (2) and (7) and that it is possible to find recurrence relations between the coefficients occurring in those asymptotic expansions.

Let α_h denote the degree of the denominator of the rational function $\psi_h(z)$ minus the degree of the numerator of that function. That the right hand side of (12) represents for large $|z|$ an asymptotic expansion, means that either the number of terms occurring on that side is finite or α_h tends to infinity for $h \to \infty$. Without loss of generality we may suppose $\alpha_0 < \alpha_1 < \alpha_2 < \ldots$, for otherwise we have only to rearrange the terms and to take, if necessary, some terms together.

For the sake of simplicity I write α instead of α_0. By hypothesis $\psi_0(z)$ tends for $|z| \to \infty$ to zero, so that $\alpha > 0$.
We have

$$\psi_0(z) = \frac{A}{z^\alpha} + \frac{B}{z^{\alpha+1}} + O\left(\frac{1}{z^{\alpha+2}}\right),$$

where $A \neq 0$ and B denote suitable chosen numbers independent of z.
If the right hand side of (12) contains more than one term, then

$$\psi_1(z) = \frac{A_1}{z^{\alpha_1}} + \ldots,$$

where $A_1 \neq 0$ denotes a suitable chosen number independent of z.

In section 2 we shall find for each function $f(z)$ which satisfies in the sector $-\pi + \epsilon < \arg z < \pi - \epsilon$ for large $|z|$ condition (12) and also the condition of smoothness formulated above, asymptotic expansions of the form

$$f(z) = A^z \alpha^{\alpha z} \left\{ \sum_{m=0}^{M-1} \frac{c_m}{\Gamma(\alpha z + \beta + m)} + O\left(\frac{1}{\Gamma(\alpha z + \beta + M)}\right) \right\} \tag{13}$$

and

$$(f(z))^{-1} = A^{-z} \alpha^{-\alpha z} \left\{ \sum_{m=0}^{M-1} \gamma_m \Gamma(\alpha z + \beta - m) + O\left(\Gamma(\alpha z + \beta - M)\right) \right\}. \tag{14}$$

Here

$$\beta = \tfrac{1}{2} - \tfrac{1}{2}\alpha - \frac{B}{A}, \tag{15}$$

if the right hand side of (12) contains only the term $\psi_0(z)$ and also if $\alpha_1 > \alpha + 1$; if however the right hand side of (12) contains more than one term and $\alpha_1 = \alpha + 1$, then

$$\beta = \tfrac{1}{2} - \tfrac{1}{2}\alpha - \frac{B + A_1}{A}. \tag{16}$$

The purpose of this paper is to show that the coefficients c_m and γ_m occurring in (13) and (14) satisfy for each positive integer m the recurrence relations

$$\left\{ \begin{aligned} c_m = & -\frac{1}{mA\alpha^{\alpha+1}} \sum_{n=0}^{m-1} \\ & c_n \{p(m+1+\alpha-n, \alpha, \beta+n, \psi_0) + p(m+1+\alpha-n, \alpha, \beta+n, \psi_1) + \ldots\} \end{aligned} \right. \tag{17}$$

and

$$\left\{ \begin{aligned} \gamma_m = & \frac{1}{mA\alpha^{\alpha+1}} \sum_{n=0}^{m-1} \\ & \gamma_n \{q(m+1+\alpha-n, \alpha, \beta+n, \psi_0) + q(m+1+\alpha-n, \alpha, \beta+n, \psi_1) + \ldots\}. \end{aligned} \right. \tag{18}$$

In these two formulas

$$p(m+1+\alpha-n,\ \alpha,\ \beta+n,\ \psi_1) \text{ and } q(m+1+\alpha-n,\ \alpha,\ \beta+n,\ \psi_1)$$

are equal to zero if $\alpha_1 > m-n+\alpha+1$; the terms $p(m+1+\alpha-n,\ \alpha,\ \beta+n,\ \psi_2)$ and $q(m+1+\alpha-n,\ \alpha,\ \beta+n,\ \psi_2)$ are equal to zero if $\alpha_2 > m-n+\alpha+1$ and so on, so that in the two formulas the coefficients of c_n and γ_n are sums of a finite number of terms.

Section 2: *Proof that the functions $f(z)$ and $\{f(z)\}^{-1}$ possess asymptotic expansions of the required form* (13) *and* (14).

We have assumed that $f(z)$ possesses in the sector $-\pi+\epsilon < \arg z < \pi-\epsilon$ for large $|z|$ the property

$$\frac{f(z+1)}{f(z)} = \frac{A}{z^\alpha} + \frac{B'}{z^{\alpha+1}} + O\left(\frac{1}{z^{\alpha+2}}\right), \tag{19}$$

where $A \neq 0$. We have

$$B' = B,$$

if the right hand side of (12) consists of only one term and also if the right hand side of (12) consists of more than one term and $\alpha_1 > \alpha+1$. If however the right hand side of (12) consists of more than one term and $\alpha_1 = \alpha+1$, then

$$B' = B + A_1.$$

The number β defined in (15) and (16) has therefore the property

$$\tfrac{1}{2}\alpha + \frac{B'}{A} = \tfrac{1}{2} - \beta, \tag{20}$$

We shall now prove, using (19) and the fact that $f(z)$ satisfies the condition of smoothness formulated in the introduction, that $f(z)$ and $\{f(z)\}^{-1}$ possess asymptotic expansions of the required form. It follows from (19) that

$$\left.\begin{aligned} \log f(z+1) - \log f(z) &= -\alpha \log z + \log A + \log\left(1 + \frac{B'}{Az} + O\left(\frac{1}{z^2}\right)\right) \\ &= -\alpha \log z + \log A + \frac{B'}{Az} + O\left(\frac{1}{z^2}\right). \end{aligned}\right\} \tag{21}$$

Furthermore

$$\begin{aligned} (z+1)\log(z+1) - z\log z &= (z+1)\left\{\log z + \frac{1}{z} - \frac{1}{2z^2} + O\left(\frac{1}{z^3}\right)\right\} - z\log z \\ &= \log z + 1 - \frac{1}{2z} + \frac{1}{z} + O\left(\frac{1}{z^2}\right), \end{aligned}$$

so that

$$\log z = (z+1)\log(z+1) - z\log z - 1 - \frac{1}{2z} + O\left(\frac{1}{z^2}\right). \tag{22}$$

Substituting this result in (21) we obtain

$$\log f(z+1) - \log f(z) = -\alpha\{(z+1)\log(z+1) - z\log z\} + \alpha + \log A$$

$$(23) \qquad + \left(\frac{\alpha}{2} + \frac{B'}{A}\right)\frac{1}{z} + O\left(\frac{1}{z^2}\right),$$

where

$$(24) \qquad \frac{1}{z} = \log(z+1) - \log z + O\left(\frac{1}{z^2}\right).$$

Thus we see that the function

$$F(z) = \log f(z) + \alpha\, z\log z - (\alpha + \log A)z - \left(\frac{\alpha}{2} + \frac{B'}{A}\right)\log z$$

satisfies the relation

$$F(z+1) - F(z) = O\left(\frac{1}{z^2}\right),$$

so that for suitably chosen K and for sufficiently large $|z|$

$$|F(z+1) - F(z)| \leqq K\,|z|^{-2}.$$

Replacing z by $z+n-1$, where n denotes a positive integer, we find

$$|F(z+n) - F(z+n-1)| \leqq K|z+n-1|^{-2},$$

so that the series

$$\sum_{n=1}^{\infty}\{F(z+n) - F(z+n-1)\}$$

converges. That means that $F(z+n)$ tends to a finite limit l as the positive integer n approaches infinity. For the proof it is necessary to show that this limit l is independent of z. It is sufficient to show that $l = \log L$, where L denotes the limit occuring in the condition of smoothness. We know that

$$(25) \quad \log f(z+n) + \alpha(z+n)\log(z+n) - (\alpha + \log A)(z+n) - \left(\frac{\alpha}{2} + \frac{B'}{A}\right)\log(z+n)$$

tends to the finite limit l, as the positive integer n approaches infinity. According to the condition of smoothness applied on $f(z)$

$$(26) \qquad \log f(z+n) - \mu(z+n)\log(z+n) - \varkappa(z+n) - \nu\log(z+n)$$

tends to the finite limit $\log L$. That is only possible if

$$\mu = -\alpha; \qquad \varkappa = \alpha + \log A \ \text{ and } \ \nu = \frac{\alpha}{2} + \frac{B'}{A},$$

so that the two functions (25) and (26) are the same. Consequently $l = \log L$. The formula

$$F(z+n) - F(z) = \sum_{h=1}^{n}\{F(z+h) - F(z+h-1)\}$$

gives therefore

$$\log L - F(z) = \sum_{h=1}^{\infty} \{F(z+h) - F(z+h-1)\}.$$

The remainder terms on the right hand sides of (21), (22), (23) and (24) possess asymptotic expansions according to ascending powers of $1/z$ with integral exponents $\geqq 2$. Consequently $F(z+1) - F(z)$ also possesses such an asymptotic expansion, which means that this function has an asymptotic expansion of the form

$$F(z+1) - F(z) \backsim C_1\left(\frac{1}{z+1} - \frac{1}{z}\right) + C_2\left(\frac{1}{(z+1)^2} - \frac{1}{z^2}\right) + \ldots.$$

In this way we find for each positive integer h

$$F(z+h) - F(z+h-1) \backsim C_1\left(\frac{1}{z+h} - \frac{1}{z+h-1}\right) + C_2\left(\frac{1}{(z+h)^2} - \frac{1}{(z+h-1)^2}\right) + \ldots$$

hence

$$\sum_{h=1}^{\infty}\{F(z+h) - F(z+h-1)\} \backsim -\frac{C_1}{z} - \frac{C_2}{z^2} - \frac{C_3}{z^3} + \ldots,$$

consequently

$$(27)\quad \left\{\begin{aligned} &\log f(z) + \alpha z \log z - (\alpha + \log A)z - \left(\frac{\alpha}{2} + \frac{B'}{A}\right)\log z = \\ &\qquad\qquad = F(z) \backsim \log L + \frac{C_1}{z} + \frac{C_2}{z^2} + \ldots \end{aligned}\right.$$

It follows from Stirling's formula that

$$(28)\qquad \log \Gamma(\alpha z + \beta) - (\alpha z + \beta - \tfrac{1}{2}) \log (\alpha z + \beta) + (\alpha z + \beta)$$

possesses an asymptotic expansion according to ascending powers of $\dfrac{1}{az+\beta}$ with integral exponents $\geqq 0$. That function also possesses therefore an asymptotic expansion according to ascending powers of $1/z$ with integral exponents $\geqq 0$. The sum of the last two terms in (28) can be written as

$$-(\alpha z + \beta - \tfrac{1}{2})\left(\log \alpha z + \log\left(1 + \frac{\beta}{\alpha z}\right)\right) + (\alpha z + \beta) =$$

$$= -\alpha z \log z + (1 - \log \alpha)\, \alpha z + (\tfrac{1}{2} - \beta) \log z + \ldots,$$

where the dots represent a power series in $1/z$. Combining this result with (27) and adding we find by (20) for the function

$$\log f(z) + \log \Gamma(\alpha z + \beta) - (\log A + \alpha \log \alpha)\, z$$

an asymptotic expansion according to ascending powers of $1/z$ with integral exponents $\geqq 0$. That means that the two functions

$$H(z) = \frac{f(z)\, \Gamma(\alpha z + \beta)}{A^z \alpha^{\alpha z}} \quad \text{and} \quad (H(z))^{-1} = \frac{A^z \alpha^{\alpha z}}{f(z)\, \Gamma(\alpha z + \beta)}$$

possess also asymptotic expansions according to ascending powers of $1/z$ with integral exponents $\geqq 0$ and that in both these expansions the constant term is $\neq 0$. We can choose therefore the constant $c_0 \neq 0$ such that

$$H(z) = c_0 + O\left(\frac{1}{z}\right);$$

then we can choose the constant c_1 such that

$$H(z) = c_0 + \frac{c_1}{\alpha z + \beta} + O\left(\frac{1}{z^2}\right),$$

and so on. In this way we find for $H(z)$ an asymptotic expansion of the form

$$H(z) \backsim c_0 + \frac{c_1}{\alpha z + \beta} + \frac{c_2}{(\alpha z + \beta)(\alpha z + \beta + 1)} + \dots,$$

which gives the required expansion (13). In the same way we find for $(H(z))^{-1}$ an asymptotic expansion of the form

$$(H(z))^{-1} \backsim \gamma_0 + \frac{\gamma_1}{\alpha z + \beta - 1} + \frac{\gamma_2}{(\alpha z + \beta - 1)(\alpha z + \beta - 2)} + \dots,$$

which yields the required expansion (14).

This completes the proof.

(To be continued)